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## Topology and its Applications

[www.elsevier.com/locate/topol](http://www.elsevier.com/locate/topol)Set-maximal selections<sup>☆</sup>Valentin Gutev<sup>a,\*</sup>, Tsugunori Nogura<sup>b</sup><sup>a</sup> School of Mathematical Sciences, University of KwaZulu-Natal, Westville Campus, Private Bag X54001, Durban 4000, South Africa<sup>b</sup> Department of Mathematics, Faculty of Science, Ehime University, Matsuyama, 790-8577, Japan

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## ABSTRACT

The present paper deals with continuous selections  $f$  for the Vietoris hyperspace  $\mathcal{F}(X)$  of all nonempty closed subsets of a space  $X$  which are maximal with respect to some closed subset  $S \subset X$ , i.e. with the property that  $f(T) \in S$  for every  $T \in \mathcal{F}(X)$  with  $T \cap S \neq \emptyset$ . This gives rise to a common point of view of several extreme-like properties studied before, it also provides an useful tool in classifying disconnectedness-like properties of spaces by continuous selections for the Vietoris hyperspace.

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## 1. Introduction

For a  $T_1$ -space  $X$ , let  $\mathcal{F}(X)$  be the set of all nonempty closed subsets of  $X$ . Usually, we endow  $\mathcal{F}(X)$  with the Vietoris topology  $\tau_V$ , and call it the Vietoris hyperspace of  $X$ . Recall that  $\tau_V$  is generated by all collections of the form

$$\langle \mathcal{V} \rangle = \left\{ S \in \mathcal{F}(X) : S \subset \bigcup \mathcal{V} \text{ and } S \cap V \neq \emptyset, \text{ whenever } V \in \mathcal{V} \right\},$$

where  $\mathcal{V}$  runs over the finite families of open subsets of  $X$ .

In the sequel, all spaces are assumed to be at least Hausdorff, while any subset  $\mathcal{D} \subset \mathcal{F}(X)$  will carry the relative Vietoris topology  $\tau_V$  as a subspace of the hyperspace  $(\mathcal{F}(X), \tau_V)$ . A map  $f : \mathcal{D} \rightarrow X$  is a *selection* for  $\mathcal{D}$  if  $f(S) \in S$  for every  $S \in \mathcal{D}$ . A selection  $f : \mathcal{D} \rightarrow X$  is *continuous* if it is continuous with respect to the relative Vietoris topology  $\tau_V$  on  $\mathcal{D}$ . For a family  $\mathcal{D} \subset \mathcal{F}(X)$ , we will use  $\nabla_{cs}[\mathcal{D}]$  to denote the set of all Vietoris continuous selections for  $\mathcal{D}$ .

This paper was motivated by recent results about continuous extreme-like selections and special points. Let us recall that a selection  $f : \mathcal{F}(X) \rightarrow X$  is *p-maximal* for a point  $p \in X$  [5,10,12] if  $f(S) = p$  for every  $S \in \mathcal{F}(X)$ , with  $p \in S$ . We say that a point  $p \in X$  is *selection-maximal* if  $\mathcal{F}(X)$  has a continuous  $p$ -maximal selection, and we say that  $X$  is *selection pointwise-maximal* if each point of  $X$  is selection maximal [12]. The following theorem summarizes some of the results in [5,10], see [5, Theorems 3.3 and 3.5] and [10, Theorem 1.4].

**Theorem 1.1.** ([5,10]) *Let  $X$  be a space, with  $\nabla_{cs}[\mathcal{F}(X)] \neq \emptyset$ . If  $X$  is selection pointwise-maximal, then it is zero-dimensional, while  $X$  is selection pointwise-maximal whenever it is zero-dimensional and first countable. If, moreover,  $X$  is separable, then it is selection pointwise-maximal if and only if it is first countable and zero-dimensional.*

Here, a space  $X$  is *zero-dimensional* if it has a base of clopen sets. A complete topological characterization of selection pointwise-maximal spaces was obtained in [12]. Another extreme-like property was introduced in [10], and studied also in [6]. The following theorem summarizes [10, Theorem 1.5] and [6, Theorem 2.1].

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**Theorem 1.2.** ([6,10]) Let  $X$  be a space, with  $\mathbb{V}_{cs}[\mathcal{F}(X)] \neq \emptyset$ . Then, the set  $\{f(X) : f \in \mathbb{V}_{cs}[\mathcal{F}(X)]\}$  is dense in  $X$  if and only if  $X$  has a clopen  $\pi$ -base. Moreover,  $X$  is totally disconnected whenever the set  $\{f(X) : f \in \mathbb{V}_{cs}[\mathcal{F}(X)]\}$  is dense in  $X$ .

Here,  $X$  is *totally disconnected* if each singleton of  $X$  is an intersection of clopen subsets of  $X$ . A family  $\mathcal{P}$  of open subsets of  $X$  is a  $\pi$ -base (sometimes, called also a *pseudobase*) for  $X$  if every nonempty open subset of  $X$  contains some nonempty member of  $\mathcal{P}$ . Concerning the second part of Theorem 1.2, the following question was suggested implicitly in [10] and posed in [11, Problem 4.2] and [14, Question 390].

**Question 1.** ([10,11,14]) Does there exist a space  $X$  which is not zero-dimensional but the set  $\{f(X) : f \in \mathbb{V}_{cs}[\mathcal{F}(X)]\}$  is dense in  $X$ ?

We are now ready to state also the main purpose of this paper. Namely, in the present paper we study extreme-like selections as those in Theorems 1.1 and 1.2 but from the perspective of a single extreme-like property. Recall that a selection  $f : \mathcal{F}(X) \rightarrow X$  is called *S-maximal* for a nonempty closed subset  $S \subset X$  if  $f(T) \in S$  for every  $T \in \mathcal{F}(X)$  with  $S \subset T$ , see [5]. Here, we are interested in a little bit stronger extreme property. Namely, we shall say that a selection  $f : \mathcal{F}(X) \rightarrow X$  is *strongly S-maximal* for a subset  $S \subset X$  if  $f(T) \in S$  for every  $T \in \mathcal{F}(X)$  with  $T \cap S \neq \emptyset$ , and we say that  $S \subset X$  is *selection strongly-maximal* if  $\mathcal{F}(X)$  has a continuous strongly  $S$ -maximal selection.

In this terminology, a selection  $f$  for  $\mathcal{F}(X)$  is *p-maximal* for a point  $p \in X$  if and only if it is (strongly)  $\{p\}$ -maximal. Also, if  $S$  is a nonempty closed subset of  $X$ , then every continuous strongly  $S$ -maximal selection is  $S$ -maximal as well, but the converse is not necessarily true. For instance, according to [15, Lemma 7.3], if  $X$  is connected and  $f \in \mathbb{V}_{cs}[\mathcal{F}(X)]$ , then  $f$  is  $p$ -maximal for some point  $p \in X$ . In this case, take any other point  $q \in X \setminus \{p\}$ , and set  $S = \{p, q\}$ . Then,  $f$  is  $S$ -maximal for this particular  $S$  but it fails to be strongly  $S$ -maximal.

In contrast to selection pointwise-maximal spaces and Theorem 1.1, the class of spaces in which every nonempty closed set is selection strongly-maximal is rather restrictive, the following theorem will be proved.

**Theorem 1.3.** Let  $X$  be a space, with  $\mathbb{V}_{cs}[\mathcal{F}(X)] \neq \emptyset$ . Then,  $X$  is a first countable space with at most one non-isolated point if and only if each  $S \in \mathcal{F}(X)$  is selection strongly-maximal.

Regarding the relationship with point-maximal selections, the following refinements of Theorem 1.1 will be proved as well.

**Theorem 1.4.** Let  $X$  be a space, with  $\mathbb{V}_{cs}[\mathcal{F}(X)] \neq \emptyset$ . Then,  $X$  is zero-dimensional if and only if for every open set  $V \subset X$  and a point  $x \in V$  there exists a selection strongly-maximal set  $S \in \mathcal{F}(X)$ , with  $x \in S \subset V$ .

**Theorem 1.5.** Let  $X$  be a space, with  $\mathbb{V}_{cs}[\mathcal{F}(X)] \neq \emptyset$ . Then,  $X$  is normal and strongly zero-dimensional if and only if for every closed set  $F \subset X$  and its neighbourhood  $V \subset X$  there exists a selection strongly-maximal set  $S \in \mathcal{F}(X)$ , with  $F \subset S \subset V$ .

Further, we demonstrate that in the present setting of Question 1, the answer is in the positive, see Example 4.4. However, the space in our example is not regular. From this perspective, the question remains open in the realm of Tychonoff spaces, see Question 3. Finally, we characterize several other disconnectedness-like properties in terms of set-maximal selections, for instance the following refinement of Theorem 1.2 will be proved.

**Theorem 1.6.** Let  $X$  be a space, with  $\mathbb{V}_{cs}[\mathcal{F}(X)] \neq \emptyset$ . Then,  $X$  is totally disconnected if and only if for every finite subset  $F \subset X$  and a point  $x \in X \setminus F$  there exists a selection strongly-maximal set  $S \in \mathcal{F}(X)$ , with  $x \in S \subset X \setminus F$ .

The paper is organized as follows. In the next section, we study special extreme-like selections for the family  $\mathcal{F}_2(X) = \{S \subset X : 1 \leq |S| \leq 2\} \subset \mathcal{F}(X)$  of at most 2-point subsets of  $X$ , such selections are naturally interrelated with the set-maximal ones. Section 3 is devoted to general facts about strongly set-maximal selections, it also contains the proof of Theorem 1.3 which is obtained as a consequence of a more general result, see Theorem 3.2. Section 4 is devoted to selections and totally disconnected spaces, in particular it contains the proof of Theorem 1.6 (see, Theorem 4.1). Section 5 deals with the proof of Theorem 1.4 and related results about selections and zero-dimensional spaces. The proof of Theorem 1.5 will be finalized in the last Section 6 of the paper which is mainly devoted to selections and strongly zero-dimensional spaces.

## 2. Weak selections and initial weak-segments

A selection  $f : \mathcal{F}_2(X) \rightarrow X$  is often called a *weak selection* for  $X$ . Every weak selection  $f$  for  $X$  defines a natural order-like relation  $\preceq_f$  on  $X$  [15] by letting for  $x, y \in X$  that  $x \preceq_f y$  if and only if  $f(\{x, y\}) = x$ . For convenience, we write that  $x \prec_f y$  if  $x \preceq_f y$  and  $x \neq y$ . The relation  $\preceq_f$  is very similar to a linear order on  $X$  in that it is both total and antisymmetric, but, unfortunately, it may fail to be transitive. Following [4], if  $B$  and  $C$  are (not necessarily nonempty) subsets of  $X$ , we

write that  $B \preceq_f C$  if  $y \preceq_f z$  for every  $y \in B$  and  $z \in C$ . In particular, we write that  $B \prec_f C$  if  $y \prec_f z$  for every  $y \in B$  and  $z \in C$  or, equivalently, if  $B \preceq_f C$  and  $B \cap C = \emptyset$ .

A starting point for the considerations in this section is the following simple observation, its verification is left to the reader.

**Proposition 2.1.** *Let  $S \subset X$ , and let  $f : \mathcal{F}(X) \rightarrow X$  be a strongly  $S$ -maximal selection. Then,  $S \prec_f X \setminus S$ .*

Motivated by this, we shall say that a subset  $A \subset X$  is an *initial weak-segment* if there exists a continuous weak selection  $f$  for  $X$  such that  $A \prec_f X \setminus A$ . In this case, to express explicitly that this is with respect to  $f$  we will often say that  $A$  is an *initial  $f$ -segment*.

To any weak selection  $f$  for  $X$  one can associate another one  $f^\perp : \mathcal{F}_2(X) \rightarrow X$  defined by  $S = \{f(S), f^\perp(S)\}$ ,  $S \in \mathcal{F}_2(X)$ . Note that if  $x, y \in X$ , then  $x \preceq_{f^\perp} y$  if and only if  $y \preceq_f x$ . That is, the  $\preceq_{f^\perp}$ -relation is reverse to the  $\preceq_f$ -one. It is well known that  $f$  is continuous if and only if so is  $f^\perp$ , see, for instance, [9, Theorem 3.5]. This implies the following simple observation.

**Proposition 2.2.** *Let  $f$  be a continuous weak selection for  $X$ . Then, a subset  $A \subset X$  is an initial  $f$ -segment if and only if  $X \setminus A$  is an initial  $f^\perp$ -segment.*

The purpose of this section is to provide a complete characterization of initial weak-segments. To this end, for a weak selection  $f$  for  $X$ , define the following  $\preceq_f$ -open intervals:

$$(\leftarrow, x)_{\preceq_f} = \{y \in X : y \prec_f x\} \quad \text{and} \quad (x, \rightarrow)_{\preceq_f} = \{y \in X : x \prec_f y\}.$$

In the same way, we define the corresponding  $\preceq_f$ -closed intervals:

$$[\leftarrow, x]_{\preceq_f} = \{y \in X : y \preceq_f x\} \quad \text{and} \quad [x, \rightarrow]_{\preceq_f} = \{y \in X : x \preceq_f y\}.$$

Finally, for points  $x, y \in X$ , we will use also the following composite  $\preceq_f$ -intervals:

$$(x, y)_{\preceq_f} = (x, \rightarrow)_{\preceq_f} \cap (\leftarrow, y)_{\preceq_f} \quad \text{and} \quad [x, y]_{\preceq_f} = [x, \rightarrow]_{\preceq_f} \cap (\leftarrow, y]_{\preceq_f}.$$

Since the relation  $\preceq_f$  is not necessarily transitive, both intervals  $(x, y)_{\preceq_f}$  and  $(y, x)_{\preceq_f}$  could be nonempty, similarly for  $[x, y]_{\preceq_f}$  and  $[y, x]_{\preceq_f}$ .

If  $f$  is a continuous weak selection for  $X$ , then the relation  $\preceq_f$  is “compatible” with the topology of  $X$ . In this case, Michael [15] demonstrated that all  $\preceq_f$ -open intervals  $(\leftarrow, x)_{\preceq_f}$  and  $(x, \rightarrow)_{\preceq_f}$ ,  $x \in X$ , are open in  $X$ . Throughout this paper, we will freely rely on this property of continuous weak selections. However, let us explicitly mention that there are weak selections which are not continuous, but all intervals of this type are open (see [9, Example 3.6] and [13, Corollary 4.2]).

**Theorem 2.3.** *Let  $X$  be a space and  $f$  be a continuous weak selection for  $X$ . If  $A \subset X$  is an initial  $f$ -segment, then  $A$  is either clopen or  $(\leftarrow, p)_{\preceq_f} \subset A \subset (\leftarrow, p]_{\preceq_f}$  for some point  $p \in X$ . Conversely, if  $A \subset X$  is clopen or there exists a point  $p \in X$  such that  $(\leftarrow, p)_{\preceq_f} \subset A \subset (\leftarrow, p]_{\preceq_f}$ , then  $A$  is an initial weak-segment.*

**Proof.** Let  $A \subset X$  be an initial  $f$ -segment. If  $A$  is not closed, then there exists a point  $p \in \bar{A} \setminus A$ . Hence,  $A \subset (\leftarrow, p)_{\preceq_f}$  because  $A \prec_f X \setminus A$ . Suppose that there is a point  $q \in (\leftarrow, p)_{\preceq_f} \setminus A$ . Then, just like before,  $A \subset (\leftarrow, q)_{\preceq_f}$  and, therefore,  $(q, \rightarrow)_{\preceq_f}$  is a neighbourhood of  $p$  such that  $(q, \rightarrow)_{\preceq_f} \cap A = \emptyset$ . However,  $p \in \bar{A}$  which is impossible, so  $A = (\leftarrow, p)_{\preceq_f}$ . If  $A$  is not open, then  $X \setminus A$  is not closed while, by Proposition 2.2,  $X \setminus A$  is an initial  $f^\perp$ -segment. Hence, by the previous arguments,  $X \setminus A = (\leftarrow, p)_{\preceq_{f^\perp}} = (p, \rightarrow)_{\preceq_f}$  and, therefore,  $A = (\leftarrow, p]_{\preceq_f}$ .

To see the converse, suppose first that  $A$  is clopen. Then, define another weak selection  $g$  for  $X$  by letting for  $x, y \in X$  that  $g(\{x, y\}) = x$  if  $x \in A$  and  $y \in X \setminus A$ , and  $g(\{x, y\}) = f(\{x, y\})$  otherwise. Since  $A$  is clopen,  $g$  is also continuous, and we now have that  $A \prec_g X \setminus A$ . So,  $A$  is an initial  $g$ -segment. Suppose finally that  $(\leftarrow, p)_{\preceq_f} \subset A \subset (\leftarrow, p]_{\preceq_f}$  for some point  $p \in X$ . To show that  $A$  is an initial weak-segment, it suffices to show that there exists a continuous weak selection  $g$  for  $X$ , with  $(\leftarrow, p]_{\preceq_f} \preceq_g [p, \rightarrow)_{\preceq_f}$ . Such a selection  $g$  can be defined by  $g(\{x, y\}) = x$  if  $x \preceq_f p \preceq_f y$ , and  $g(\{x, y\}) = f(\{x, y\})$  otherwise. The continuity of  $g$  in pairs  $\{x, y\}$ , with  $x \neq p \neq y$ , follows easily by the continuity of  $f$ . Take a point  $x \in X$ , with  $x \prec_f p$ . Then, by [9, Theorem 3.1], there are open sets  $U, V \subset X$  such that  $x \in U$ ,  $p \in V$  and  $U \prec_f V$ . We now have that  $U \prec_g V$ . Indeed, take a point  $s \in U$  and  $t \in V$ . If  $p \preceq_f t$ , then, by the definition of  $g$ , we get that  $s \prec_g t$  because  $s \in U \subset (\leftarrow, p)_{\preceq_f}$ . If  $t \preceq_f p$ , then  $g(\{s, t\}) = f(\{s, t\}) = s$ . Hence, we get again that  $s \prec_g t$ . Thus,  $U \prec_g V$  and, by [9, Theorem 3.1],  $g$  must be continuous at  $\{x, p\}$ . Since the case  $p \prec_f x$  is completely analogous, the proof is completed.  $\square$

### 3. Selection strongly-maximal sets

**Proposition 3.1.** *Let  $S \subset X$ , and let  $f : \mathcal{F}(X) \rightarrow X$  be a continuous strongly  $S$ -maximal selection. Then  $f$  is also strongly  $\bar{S}$ -maximal.*

**Proof.** Let  $\bar{S} \setminus S \neq \emptyset$  and let  $T \in \mathcal{F}(X)$  be such that  $T \cap S = \emptyset \neq T \cap \bar{S}$ . If  $U$  is a neighbourhood of  $f(T)$ , then there exists a finite family  $\mathcal{V}$  of open subsets of  $X$  such that  $T \in \langle \mathcal{V} \rangle$  and  $f(\langle \mathcal{V} \rangle) \subset U$ . Since  $\emptyset \neq T \cap \bar{S} \subset \bigcup \mathcal{V}$ , there now exists an  $F \in \langle \mathcal{V} \rangle$  such that  $F \cap S \neq \emptyset$ . Hence,  $f(F) \in S \cap U$  which implies that  $f(T) \in \bar{S}$ .  $\square$

Recall that a subset  $S \subset X$  was called *selection strongly-maximal* if  $\mathcal{F}(X)$  has a continuous strongly  $S$ -maximal selection. In our next considerations, in view of Proposition 3.1, we may concentrate only on closed selection strongly-maximal sets. Now, by analogy with selection pointwise-maximal spaces, we shall say that  $X$  is *selection strongly  $\mathcal{D}$ -maximal* for some nonempty family  $\mathcal{D} \subset \mathcal{F}(X)$  if every  $S \in \mathcal{D}$  is selection strongly-maximal. The purpose of this section is to establish the following slight generalization of Theorem 1.6.

**Theorem 3.2.** For a space  $X$ , with  $\mathbb{V}_{\text{cs}}[\mathcal{F}(X)] \neq \emptyset$ , the following are equivalent:

- (a)  $X$  is selection strongly  $\mathcal{F}(X)$ -maximal.
- (b)  $X$  is selection strongly  $[X]^2$ -maximal, where  $[X]^2 = \{S \subset X: |S| = 2\}$ .
- (c)  $X$  is a first countable space with at most one non-isolated point.

To prepare for the proof of Theorem 3.2, we proceed with several observations about selection strongly-maximal sets.

**Proposition 3.3.** Let  $X$  be a space, with  $\mathbb{V}_{\text{cs}}[\mathcal{F}(X)] \neq \emptyset$ . Then, every clopen subset  $S \subset X$  is selection strongly-maximal.

**Proof.** Take a selection  $g \in \mathbb{V}_{\text{cs}}[\mathcal{F}(X)]$ , and define  $f: \mathcal{F}(X) \rightarrow X$  by letting for  $F \in \mathcal{F}(X)$  that  $f(F) = g(F \cap S)$  if  $F \cap S \neq \emptyset$ , and  $f(F) = g(F)$  otherwise. Then,  $f \in \mathbb{V}_{\text{cs}}[\mathcal{F}(X)]$  because  $S$  is clopen, and clearly  $f$  is strongly  $S$ -maximal.  $\square$

**Proposition 3.4.** Let  $S \in \mathcal{F}(X)$ , and let  $f \in \mathbb{V}_{\text{cs}}[\mathcal{F}(X)]$  be a strongly  $S$ -maximal selection. If  $S$  is not open, then there exists a point  $p \in X$  such that

- (a)  $S = (\leftarrow, p]_{\preccurlyeq_f}$ ,
- (b)  $\mathcal{F}(Z)$  has a continuous  $p$ -maximal selection, where  $Z = [p, \rightarrow)_{\preccurlyeq_f}$ .

**Proof.** The statement of (a) follows by Proposition 2.1 and Theorem 2.3. To show (b), define a continuous selection  $g$  for  $\mathcal{F}(Z)$  by  $g = f \upharpoonright \mathcal{F}(Z)$ . Then, whenever  $F \in \mathcal{F}(Z)$  and  $p \in F$ , we have that  $g(F) = f(F) \in S \cap F = \{p\}$ . That is,  $g(F) = p$  which completes the proof.  $\square$

In our next proposition, a selection  $f: \mathcal{F}(Y) \rightarrow Y$  is called  *$p$ -minimal* [5] provided  $f(F) = p$  if and only if  $F = \{p\}$ .

**Proposition 3.5.** Let  $Y, Z \in \mathcal{F}(X)$  be such that  $Y \cup Z = X$  and  $Y \cap Z = \{p\}$  for some point  $p \in X$ . Suppose that  $\mathcal{F}(Y)$  has a continuous  $p$ -minimal selection, and  $\mathcal{F}(Z)$  has a continuous  $p$ -maximal one. Then,  $Y$  is a selection strongly-maximal set.

**Proof.** The proof relies on a construction in [5, Lemma 6.4]. Let us briefly recall this construction. Take a continuous  $p$ -minimal selection  $g$  for  $\mathcal{F}(Y)$ , and a continuous  $p$ -maximal selection  $h$  for  $\mathcal{F}(Z)$ . Consider the family

$$\mathcal{F}_X(Y) = \{F \in \mathcal{F}(X): F \cap Y \neq \emptyset\}.$$

Then,  $\mathcal{F}_X(Y)$  is a  $\tau_V$ -closed subset of  $\mathcal{F}(X)$  such that

$$\mathcal{F}_X(Y) \cup \mathcal{F}(Z) = \mathcal{F}(X),$$

$$\mathcal{F}_X(Y) \cap \mathcal{F}(Z) = \{F \in \mathcal{F}(Z): p \in F\}.$$

Next, define a selection  $f: \mathcal{F}(X) \rightarrow X$  by  $f \upharpoonright \mathcal{F}_X(Y) = g \circ \varphi$  and  $f \upharpoonright \mathcal{F}(Z) = h$ , where  $\varphi: \mathcal{F}_X(Y) \rightarrow \mathcal{F}(Y)$  is defined by  $\varphi(F) = (F \cap Y) \cup \{p\}$ ,  $F \in \mathcal{F}_X(Y)$ . Since  $g$  is  $p$ -minimal and  $h$  is  $p$ -maximal, the definition of  $f$  is correct. Also, according to the proof of [5, Lemma 6.4],  $f$  is continuous. If  $F \in \mathcal{F}(X)$  and  $F \cap Y \neq \emptyset$ , then  $F \in \mathcal{F}_X(Y)$  and, by the definition of  $f$ , we have that

$$f(F) = g(\varphi(F)) = g((F \cap Y) \cup \{p\}) \in Y \cup \{p\} = Y.$$

That is,  $f$  is strongly  $Y$ -maximal.  $\square$

**Proposition 3.6.** Let  $Y$  be a space, with  $\mathbb{V}_{\text{cs}}[\mathcal{F}(Y)] \neq \emptyset$ , and let  $p \in Y$  be such that  $\{p\}$  is a countable intersection of clopen subsets of  $Y$ . Then,  $\mathcal{F}(Y)$  has a continuous  $p$ -minimal selection.

**Proof.** Take a decreasing clopen family  $\{V_n: n < \omega\}$  such that  $V_0 = Y$  and  $\{p\} = \bigcap \{V_n: n < \omega\}$ . Next, for every  $F \in \mathcal{F}(Y)$ , with  $F \neq \{p\}$ , set

$$n(F) = \min\{n < \omega: F \cap S_n \neq \emptyset\},$$

where  $S_n = V_n \setminus V_{n+1}$ ,  $n < \omega$ . Finally, take a selection  $f \in \mathbb{V}_{cs}[\mathcal{F}(Y)]$ , and define another one  $g: \mathcal{F}(Y) \rightarrow Y$  by letting for  $F \in \mathcal{F}(Y)$  that  $g(F) = p$  if  $F = \{p\}$ , and  $g(F) = f(F \cap S_{n(F)})$  otherwise. The definition is correct because each  $S_n$  is a clopen set, and clearly  $g$  is  $p$ -minimal. We are going to show that  $g$  is continuous. Take an  $F \in \mathcal{F}(Y)$ , with  $F \neq \{p\}$ , and a neighbourhood  $U$  of  $g(F)$ . Since  $h = f \upharpoonright \mathcal{F}(S_{n(F)})$  is continuous, there exists a finite family  $\mathcal{W}_0$  of nonempty open subsets of  $S_{n(F)}$  such that  $F \cap S_{n(F)} \in \langle \mathcal{W}_0 \rangle \subset h^{-1}(U)$ . Let  $W_0 = V_{n(F)+1} \cup (\bigcup \mathcal{W}_0)$ , and let  $\mathcal{W} = \{W_0\} \cup \mathcal{W}_0$ . Then,  $\langle \mathcal{W} \rangle$  is a  $\tau_V$ -neighbourhood of  $F$ , with  $g(\langle \mathcal{W} \rangle) \subset U$ . Indeed,  $T \in \langle \mathcal{W} \rangle$  implies  $T \cap S_{n(F)} \neq \emptyset$  and  $T \subset V_{n(F)}$ , so  $n(T) = n(F)$ . On the other hand,  $T \cap S_{n(F)} \in \langle \mathcal{W}_0 \rangle$  and therefore  $g(T) = f(T \cap S_{n(T)}) \in U$ .  $\square$

We conclude the preparation for the proof of Theorem 3.2 with the following consequence which provides a complete characterization of selection strongly-maximal subsets in the realm of first countable zero-dimensional spaces.

**Corollary 3.7.** Suppose that  $X$  is a first countable zero-dimensional space, with  $\mathbb{V}_{cs}[\mathcal{F}(X)] \neq \emptyset$ . For a closed set  $S \subset X$ , the following are equivalent:

- (a)  $S$  is selection strongly-maximal.
- (b)  $S$  is an initial weak-segment.
- (c)  $|S \cap \overline{X \setminus S}| \leq 1$ .

**Proof.** The implication (a)  $\Rightarrow$  (b) follows by Proposition 2.1. The implication (b)  $\Rightarrow$  (c) follows by Theorem 2.3. To see finally that (c)  $\Rightarrow$  (a), by Proposition 3.3, we may suppose that  $S$  is not (cl)open. Next, for convenience, set  $T = \overline{X \setminus S}$ . Then,  $S \cup T = X$  and  $S \cap T = \{p\}$  for some point  $p \in X$ . Since  $T$  is first countable and zero-dimensional because so is  $X$ , by Theorem 1.1,  $\mathcal{F}(T)$  has a continuous  $p$ -maximal selection. By the same reason and Proposition 3.6,  $\mathcal{F}(S)$  has a continuous  $p$ -minimal selection. Then, by Proposition 3.5,  $\mathcal{F}(X)$  has a continuous strongly  $S$ -maximal selection.  $\square$

**Proof of Theorem 3.2.** The implication (a)  $\Rightarrow$  (b) is obvious. As for (b)  $\Rightarrow$  (c), take a non-isolated point  $p \in X$ , another point  $q \in X \setminus \{p\}$ , and let  $S = \{p, q\}$ . Since  $S$  is not open and  $\mathcal{F}(X)$  has a continuous strongly  $S$ -maximal selection  $f$ , by Proposition 3.4, we now have that  $S = (\leftarrow, p]_{\preccurlyeq_f}$  or  $S = (\leftarrow, q]_{\preccurlyeq_f}$ . Consequently, one of the points  $p$  or  $q$  must be isolated in  $X$ , so  $q$  is isolated in  $X$ . Thus,  $X$  has at most one non-isolated point. If  $X$  is a discrete space, then clearly it is first countable. Suppose that  $X$  has a non-isolated point  $p$ . Take another point  $q \in X \setminus \{p\}$ , and let  $f \in \mathbb{V}_{cs}[\mathcal{F}(X)]$  be a strongly  $S$ -maximal selection for  $S = \{p, q\}$ . Since  $q$  is an isolated point of  $X$ , we may define another selection  $g \in \mathbb{V}_{cs}[\mathcal{F}(X)]$  by letting for  $T \in \mathcal{F}(X)$  that  $g(T) = f(T \setminus \{q\})$  if  $T \neq \{q\}$  and  $g(\{q\}) = q$ . Then,  $g$  will be  $p$ -maximal and, by [5, Corollary 4.5],  $p$  must be a  $G_\delta$ -point of  $X$ . Finally, by the same reason, but now relying on [12, Theorem 1.1],  $X$  must be first countable at  $p$ . This demonstrates (c). To show finally that (c)  $\Rightarrow$  (a), suppose that  $X$  is a first countable space with at most one non-isolated point. Then,  $X$  is zero-dimensional. Take a closed subset  $S \subset X$ , and set  $T = \overline{X \setminus S}$ . Since  $X$  has at most one non-isolated point we have  $|S \cap T| \leq 1$ . Hence, by Corollary 3.7,  $\mathcal{F}(X)$  has a continuous strongly  $S$ -maximal selection.  $\square$

**Question 2.** Let  $X$  be a space, with  $\mathbb{V}_{cs}[\mathcal{F}(X)] \neq \emptyset$ , and let  $p \in X$  be a  $G_\delta$ -point such that  $X = (\leftarrow, p]_{\preccurlyeq_f}$  for some continuous weak selection  $f$  for  $X$ . Is it true that  $\mathcal{F}(X)$  has a continuous  $p$ -minimal selection?

#### 4. Selections and totally disconnected spaces

In this section, we finalize the proof of Theorem 1.6. In fact, this theorem is an immediate consequence of the following more general result. In what follows,  $\mathcal{C}(X) = \{S \in \mathcal{F}(X): S \text{ is compact}\}$ .

**Theorem 4.1.** For a space  $X$ , with  $\mathbb{V}_{cs}[\mathcal{F}(X)] \neq \emptyset$ , the following are equivalent:

- (a) for every  $T \in [X]^2$  and  $x \in X \setminus T$  there exists a selection strongly-maximal set  $S \in \mathcal{F}(X)$ , with  $x \in S \subset X \setminus T$ ;
- (b)  $X$  is totally disconnected;
- (c) for every  $T \in \mathcal{C}(X)$  and  $x \in X \setminus T$  there exists a selection strongly-maximal set  $S \in \mathcal{F}(X)$ , with  $x \in S \subset X \setminus T$ .

To prepare for the proof of Theorem 4.1, we first demonstrate it in the case of initial weak-segments.

**Lemma 4.2.** For a space  $X$  which has a continuous weak selection, the following are equivalent:

- (a) for every  $T \in [X]^2$  and a point  $x \in X \setminus T$ , there exists an initial weak-segment  $A \subset X$ , with  $x \in A \subset X \setminus T$ ;
- (b)  $X$  is totally disconnected;
- (c) for every  $T \in \mathcal{C}(X)$  and a point  $x \in X \setminus T$ , there exists an initial weak-segment  $A \subset X$ , with  $x \in A \subset X \setminus T$ .

**Proof.** (a)  $\Rightarrow$  (b). According to [10, Theorem 4.1], it suffices to show that each (connected) component of  $X$  is a singleton provided  $X$  is as in (a). Take a connected subset  $C \subset X$  and suppose that it has at least two distinct points  $y, z \in C$ . Also, take a continuous weak selection  $g$  for  $X$ , with  $y \prec_g z$ . According to [7, Lemma 2.5], there exists a point  $x \in (y, z)_{\prec_g} \subset C$ , and we may apply (a) with  $T = \{y, z\}$  and  $x$ . Hence, there exists an initial weak-segment  $A \subset X$  such that  $x \in A \subset X \setminus T$ . Then, by definition,  $A \prec_f X \setminus A$  for some continuous weak selection  $f$  for  $X$ . However, by a result of Michael [15, Lemma 7.2] (see, also, Eilenberg [3]),  $C$  has exactly 2 continuous weak selections. Hence, we now have that  $f \upharpoonright_{\mathcal{F}_2(C)} = g \upharpoonright_{\mathcal{F}_2(C)}$  or  $f \upharpoonright_{\mathcal{F}_2(C)} = g^\perp \upharpoonright_{\mathcal{F}_2(C)}$ . That is, we get that  $y \prec_f x \prec_f z$  or  $z \prec_f x \prec_f y$ , while  $x \prec_f y$  and  $x \prec_f z$  because  $A \prec_f X \setminus A$ . This is clearly impossible, which implies that  $C$  must be a singleton. Consequently,  $X$  must be totally disconnected. The implication (b)  $\Rightarrow$  (c) follows by Theorem 2.3 because in a totally disconnected space every two disjoint compact sets can be separated by a clopen set. The implication (c)  $\Rightarrow$  (a) is obvious.  $\square$

**Proof of Theorem 4.1.** The implication (a)  $\Rightarrow$  (b) follows by Proposition 2.1 and Lemma 4.2. The implication (b)  $\Rightarrow$  (c) follows by Proposition 3.3 because in a totally disconnected space every two disjoint compact sets can be separated by a clopen set. Since the implication (c)  $\Rightarrow$  (a) is obvious, the proof is completed.  $\square$

Related to Theorem 1.2, let us also mention the following simple observation.

**Proposition 4.3.** For a space  $X$ , with  $\mathbb{V}_{cs}[\mathcal{F}(X)] \neq \emptyset$ , the following are equivalent:

- (a) every nonempty open subset of  $X$  contains a nonempty selection strongly-maximal subset of  $X$ ;
- (b) the set  $\{f(X) : f \in \mathbb{V}_{cs}[\mathcal{F}(X)]\}$  is dense in  $X$ ;
- (c)  $X$  has a clopen  $\pi$ -base.

**Proof.** The implication (a)  $\Rightarrow$  (b) is obvious, that of (b)  $\Rightarrow$  (c) follows by [6, Theorem 2.1], while (c)  $\Rightarrow$  (a) follows by Proposition 3.3.  $\square$

We conclude this section with an example related to Question 1.

**Example 4.4.** There exists a space  $X$  which is not zero-dimensional but the set  $\{f(X) : f \in \mathbb{V}_{cs}[\mathcal{F}(X)]\}$  is dense in  $X$ .

**Proof.** Let  $\mathcal{C}$  be the standard Cantor set in the interval  $[0, 1]$  and, for convenience, let  $p = 1 \in \mathcal{C}$ . Also, let  $\leq$  be the linear order on  $\mathcal{C}$  inherit from the usual linear order on  $[0, 1]$ . We define our space  $X$  by modifying the topology of  $\mathcal{C}$  in the point  $p$ . Namely, take an infinite discrete subset  $D \subset \mathcal{C} \setminus \{p\}$  so that its closure in  $\mathcal{C}$  is  $D \cup \{p\}$ , say  $D$  is a sequence in  $\mathcal{C} \setminus \{p\}$  convergent to  $p$ . Next, define another topology on  $\mathcal{C}$  in which a subset  $U \subset \mathcal{C}$  is open if and only if  $U$  is open in  $\mathcal{C}$  and  $p \notin U$ , or  $p \in U$  and  $U = V \setminus D$  for some open set  $V$  in  $\mathcal{C}$ . Call the resulting topological space as  $X$ . In fact, the topology of  $X$  is obtained from the topology of  $\mathcal{C}$  by making  $D$  to be a closed discrete subset of  $X$ . Since the open interval topology on  $X$  generated by the linear order  $\leq$  coincides with the topology on  $\mathcal{C}$ , it will be a coarser topology. On the other hand, the closure of  $D$  in  $\mathcal{C}$  is  $D \cup \{p\}$ . Hence  $\{S \cup \{p\} : S \in \mathcal{F}(X)\} \subset \mathcal{F}(\mathcal{C})$  and, therefore,  $g(S) = \min_{\leq}(S \cup \{p\}) = \min_{\leq} S$ ,  $S \in \mathcal{F}(X)$ , defines a selection  $g$  for  $\mathcal{F}(X)$ . According to [15, Lemma 7.5.1],  $g$  is continuous, i.e.  $\mathbb{V}_{cs}[\mathcal{F}(X)] \neq \emptyset$ .

Keeping in mind this, we are going to show that  $X$  is as required. Take a neighbourhood  $U$  of  $p$  in  $X$ . Then,  $U = V \setminus D$  for some open set  $V$  in  $\mathcal{C}$ . So, there is a point  $q \in V \cap D$ . Take a neighbourhood  $W$  of  $q$  in  $X$  such that  $p \notin W$  and  $W \cap D = \{q\}$ . Then,  $W$  is open in  $\mathcal{C}$  and, therefore,  $W \cap U = W \cap (V \setminus D) \neq \emptyset$  because  $\mathcal{C}$  has no isolated points. This implies that the closure of  $U$  in  $X$  must contain  $q$  and, in particular, will intersect  $D$ . That is,  $X$  cannot be regular, so it will also fail to be zero-dimensional.

To show finally that  $\{f(X) : f \in \mathbb{V}_{cs}[\mathcal{F}(X)]\}$  is dense in  $X$ , let us observe that  $X$  has a clopen  $\pi$ -base. Namely, every nonempty open subset  $W$  of  $X$  will contain a nonempty open subset  $U$ , with  $p \notin U$ . Such an  $U$  will be also open in  $\mathcal{C}$ , hence it will contain a nonempty clopen subset of  $\mathcal{C}$  which will be clearly clopen in  $X$  as well. Thus,  $X$  has a clopen  $\pi$ -base and the statement follows by Theorem 1.2 because  $\mathbb{V}_{cs}[\mathcal{F}(X)] \neq \emptyset$ .  $\square$

Motivated by Example 4.4, we refine [11, Problem 4.2] and [14, Question 390] in a more natural setting relating them to zero-dimensionality rather than regularity.

**Question 3.** Does there exist a Tychonoff space  $X$  which is not zero-dimensional but the set  $\{f(X) : f \in \mathbb{V}_{cs}[\mathcal{F}(X)]\}$  is dense in  $X$ ?

## 5. Selections and zero-dimensional spaces

In this section, we first finalize the proof of Theorem 1.4.

**Proof of Theorem 1.4.** Suppose that for every open  $U \subset X$  and a point  $x \in U$ , there exists a closed selection strongly-maximal set  $S \subset X$ , with  $x \in S \subset U$ . By Theorem 1.6,  $X$  must be totally disconnected. In order to show that  $X$  is also

zero-dimensional, take an open set  $V \subset X$  and a point  $x \in V$ . Then, there exists a closed set  $S \subset X$  and a strongly  $S$ -maximal selection  $f \in \mathbb{V}_{cs}[\mathcal{F}(X)]$  such that  $x \in S \subset V$ . If  $S$  is also open, then  $S$  is a clopen set, with  $x \in S \subset V$ . Suppose that  $S$  is not open. Then, by Proposition 3.4, there exists a point  $p \in X$  such that  $S = (\leftarrow, p]_{\preceq_f}$  and  $\mathcal{F}(Z)$  has a continuous  $p$ -maximal selection  $g$ , where  $Z = [p, \rightarrow)_{\preceq_f}$ . Let  $U = V \cap Z$ , and let us show that  $U$  contains a clopen in  $Z$  subset  $G \subset Z$  such that  $p \in G \subset U$ . Indeed, if  $U = Z$ , take  $G = Z$ . Otherwise,  $T = Z \setminus U$  will be a nonempty closed subset of  $Z$  such that  $p \notin T$ . Hence,  $g(T) = q \neq p$ . Now, recall that  $X$  is totally disconnected, hence so is  $Z$  and we may repeat some of the arguments in the proof of [10, Theorem 4.1]. Namely, take a clopen in  $Z$  neighbourhood  $H$  of  $q$ , with  $p \notin H$ . Let  $\mathcal{M} \subset g^{-1}(H)$  be a chain which is a maximal with respect to the usual set-theoretical inclusion and  $T \in \mathcal{M}$ . According to the proof of [10, Theorem 4.1] (see, also, [1,2,8]),  $\mathcal{M}$  has a maximal element  $M$  which is a clopen subset of  $Z$  because  $g^{-1}(H)$  is  $\tau_V$ -clopen in  $\mathcal{F}(Z)$ . Then,  $T \subset M$  while  $p \notin M$  because  $g$  is a  $p$ -maximal selection for  $\mathcal{F}(Z)$  and  $p \notin H$ . We can now take  $G = Z \setminus M$ .

We complete this implication by showing that  $O = S \cup G$  is a clopen subset of  $X$  where  $G$  is a clopen subset of  $Z$ , with  $p \in G \subset V$ . This is almost obvious. Namely,  $O$  is closed as a union of two closed sets. Since  $G$  is open in  $Z$ , there exists an open set  $E$  in  $X$  such that  $E \cap Z = G$ . Then,  $O$  is open in  $X$  because  $O = (\leftarrow, p)_{\preceq_f} \cup E$ . Thus,  $X$  must be zero-dimensional. Since the converse follows by Proposition 3.3, the proof is completed.  $\square$

In the same way, we also have the following characterization of zero-dimensional spaces in terms of special continuous weak selections.

**Theorem 5.1.** *Let  $X$  be a space which has a continuous weak selection. Then,  $X$  is zero-dimensional if and only if for every open  $U \subset X$  and a point  $x \in U$ , there exists an open initial weak-segment  $A \subset X$ , with  $x \in A \subset U$ .*

**Proof.** Suppose that for every open  $U \subset X$  and a point  $x \in U$ , there exists an open initial weak-segment  $A \subset X$ , with  $x \in A \subset U$ . Then, by Lemma 4.2,  $X$  must be totally disconnected. Keeping in mind this, take an open  $U \subset X$  and a point  $x \in U$ . By hypothesis, there exists an open initial weak-segment  $A \subset X$  such that  $x \in A \subset U$ . By definition,  $A \prec_f X \setminus A$  for some continuous weak selection  $f$  for  $X$ . If  $A$  is not closed, then, by Theorem 2.3,  $A = (\leftarrow, p)_{\preceq_f}$  for some point  $p \in X$ . Since  $X$  is totally disconnected, we may now take a clopen set  $G \subset X$  such that  $p \in G$  and  $x \notin G$ . Thus, we get a clopen set

$$B = (\leftarrow, p)_{\preceq_f} \setminus G = (\leftarrow, p]_{\preceq_f} \setminus G$$

with the property that  $x \in B \subset U$ . If  $A$  is closed, then it is itself a clopen set with the same property as  $B$ . This implies that  $X$  is zero-dimensional. Since the converse follows by Theorem 2.3, the proof is completed.  $\square$

A word should be said about the difference between Theorems 1.4 and 5.1, and the requirement in Theorem 5.1 the initial weak-segment to be open.

**Example 5.2.** There exists a space  $X$  which is not zero-dimensional but for every open  $U \subset X$  and a point  $x \in U$ , there exists an initial weak-segment  $A \subset X$ , with  $x \in A \subset U$ .

**Proof.** Let  $X$  be the space of Example 4.4, and let  $p = 1 \in X$  be as in the proof of that example. Then,  $X$  is not zero-dimensional. Let us show that it is as required. Take an open  $U \subset X$  and a point  $x \in U$ . If  $x \neq p$ , according to the definition of the topology of  $X$ , there is a clopen set  $A \subset X$ , with  $x \in A \subset U$ . By Theorem 2.3,  $A$  is an initial weak-segment. If  $x = p$ , define a continuous weak selection  $g$  for  $X$  by  $g(\{y, z\}) = \max_{\leq} \{y, z\}$ , where  $\leq$  is the usual order on  $X$  as a subset of  $[0, 1]$ . Then,  $x = p \in (\leftarrow, p]_{\preceq_g} = \{p\} \subset U$  while, by Theorem 2.3,  $A = (\leftarrow, p]_{\preceq_g}$  is an initial weak-segment.  $\square$

Finally, let us also remark on the difference between selection-maximal points and sets, i.e. on the difference between Theorems 1.1 and 1.4. In this regard, we have the following result.

**Corollary 5.3.** *Let  $X$  be a selection pointwise-maximal space. Then, every nonempty closed initial weak-segment is selection strongly-maximal.*

**Proof.** Take a point  $p \in X$ , and a continuous weak selection  $f$  for  $X$ . According to Theorem 2.3 and Proposition 3.3, it suffices to show that  $S = (\leftarrow, p]_{\preceq_f}$  is selection strongly-maximal provided  $p$  is a non-isolated point for both  $S$  and  $T = [p, \rightarrow)_{\preceq_f}$ . In this case,  $p$  is a cut point in the sense of [12], while, by hypothesis,  $\mathcal{F}(X)$  has a continuous  $p$ -maximal selection. Then, according to [12, Theorem 3.1],  $X$  has a countable clopen base at  $p$ . It now follows by Proposition 3.6 that  $\mathcal{F}(S)$  has a continuous  $p$ -minimal selection, and, by Theorem 1.1, that  $\mathcal{F}(T)$  has a continuous  $p$ -maximal selection. Hence, Proposition 3.5 completes the proof.  $\square$

On the other hand, the converse of Corollary 5.3 is not necessarily true.

**Example 5.4.** There exists a zero-dimensional space  $X$ , with  $\mathbb{V}_{cs}[\mathcal{F}(X)] \neq \emptyset$ , such that every nonempty closed initial weak-segment is selection strongly-maximal, but  $X$  is not selection pointwise-maximal.

**Proof.** For instance, take  $X$  to be the space obtained from the topological sum of the ordinal spaces  $\omega + 1$  and  $\omega_1 + 1$  by identifying  $\omega$  and  $\omega_1$  into a single point  $p \in X$ . Then,  $X$  is not selection pointwise-maximal because  $p$  is a cut point of  $X$  in the sense of [12], but  $X$  is not first countable at  $p$ . Hence,  $\mathcal{F}(X)$  has no continuous  $p$ -maximal selection, see [12, Theorem 3.1]. However, every nonempty closed initial weak-segment of  $X$  is selection strongly-maximal because if  $S, T \in \mathcal{F}(X)$  and  $q \in X$  are such that  $S \cap T = \{q\}$  and  $S \cup T = X$ , then at least one of the sets  $S$  or  $T$  must be first countable at  $q$ . This is obvious when  $q \neq p$  because in this case  $X$  itself is first countable at  $q$ . Hence, we can use Theorem 1.1 and Propositions 3.5 and 3.6 to derive that  $\mathcal{F}(X)$  will have both continuous strongly  $S$ - and strongly  $T$ -maximal selections. If  $q = p$ , we claim that

$$S \cap \omega_1 \text{ is countable and } T \cap \omega \text{ is finite.} \quad (5.1)$$

Suppose that this is true. Then,  $S \cap \omega_1$  will be a clopen subset of  $S$  and, therefore,  $\mathcal{F}(S)$  will have both a continuous  $p$ -maximal selection and a continuous  $p$ -minimal one because so does  $\mathcal{F}(\omega + 1)$  with respect to  $p = \omega$ . In the same way,  $T \cap \omega$  will be clopen in  $T$  and  $\mathcal{F}(T)$  will have both a continuous  $p$ -maximal selection and a continuous  $p$ -minimal one because so does  $\mathcal{F}(\omega_1 + 1)$  with respect to  $p = \omega_1$ . Hence, by Proposition 3.5,  $S$  and  $T$  are both selection strongly-maximal sets.

Thus, to finish the proof, it only remains to show that (5.1) holds. To this end, suppose that  $S = (\leftarrow, p]_{\preceq_f}$  and  $T = [p, \rightarrow)_{\preceq_f}$  for some continuous weak selection  $f$  for  $X$ , but both  $S$  and  $T$  are not first countable at  $p$ . Then, for every  $n < \omega$  there are ordinals  $\alpha_n, \beta_n < \omega_1$  such that  $\alpha_n < \beta_n < \alpha_{n+1}$ ,  $\alpha_n \in S$  and  $\beta_n \in T$ . In this case,

$$\gamma = \sup\{\alpha_n : n < \omega\} = \sup\{\beta_n : n < \omega\} < \omega_1,$$

but  $\gamma \in S \cap T = \{p\}$  which is clearly impossible. Hence, one of these sets must be first countable at  $p$ , say so is  $S$ . This implies that  $S$  must be countable, otherwise we will get that  $\omega_1$  has a countable cofinality but it is impossible because  $\omega_1$  is regular. So,  $S \cap \omega_1$  is countable. Suppose finally that  $A = T \cap \omega$  is infinite. Then,  $A \subset T \setminus \{p\}$  is a countable set, with  $p \in \bar{A}$ , and, according to [5, Theorem 4.1],  $p$  will be a  $G_\delta$ -point in  $T = [p, \rightarrow)_{\preceq_f}$ . However,  $T$  is a zero-dimensional compact space, hence  $T$  will be also first countable at  $p = \omega_1$ . This is impossible because  $X$  is not first countable at  $p$ . Thus,  $T \cap \omega$  is finite.  $\square$

Motivated by this, we have the following question.

**Question 4.** Let  $X$  be a (zero-dimensional) space in which every nonempty closed initial weak-segment is selection strongly-maximal. Take a point  $p \in X$  and a continuous weak selection  $f$  for  $X$ . Is it true that one of the sets  $(\leftarrow, p]_{\preceq_f}$  or  $[p, \rightarrow)_{\preceq_f}$  will be first countable at  $p$ ?

## 6. Selections and strongly zero-dimensional spaces

Recall that a space  $X$  is *strongly zero-dimensional* if its covering dimension is zero. We conclude this paper with the proof of Theorem 1.5, and some related results.

**Proof of Theorem 1.5.** It suffices to show that  $X$  is normal and strongly zero-dimensional if it has the property stated in that theorem. In this case, by Theorem 1.4,  $X$  must be zero-dimensional. Take a closed set  $F \subset X$  and an open set  $V \subset X$ , with  $F \subset V$ . By hypothesis, there exists a selection strongly-maximal set  $S \subset X$  such that  $F \subset S \subset V$ . Then, there exists a clopen subset  $G \subset X$ , with  $F \subset G \subset V$ . Indeed, if  $S$  is open, we can take  $G = S$ . If  $S$  is not open, then, by Proposition 3.4, there exists a point  $p \in S$  such that  $S \setminus \{p\}$  is open. Since  $X$  is zero-dimensional, there now exists a clopen subset  $H \subset X$  such that  $p \in H \subset V$ . Then,  $G = S \cup H$  is a clopen subset of  $X$ , with  $F \subset G \subset V$ .  $\square$

Just like before, we have also a characterization of strongly zero-dimensional spaces in terms of initial weak-segments.

**Theorem 6.1.** Let  $X$  be a space which has a continuous weak selection. Then,  $X$  is normal and strongly zero-dimensional if and only if for every open set  $V \subset X$  and a closed set  $T \subset X$ , with  $T \subset V$ , there exists an open initial weak-segment  $A \subset X$  such that  $T \subset A \subset V$ .

**Proof.** The proof follows that of Theorem 5.1. In fact, it only suffices to show that  $X$  is normal and strongly zero-dimensional if it has the property stated in this lemma. To this end, first observe that, by Theorem 5.1,  $X$  must be zero-dimensional. Take an open set  $V \subset X$  and a closed set  $T \subset X$ , with  $T \subset V$ . By hypothesis, there exists a continuous weak selection  $f$  for  $X$  and an open initial  $f$ -segment  $A \subset X$  such that  $T \subset A \subset V$ . Suppose that  $A$  is not closed. Then, by Theorem 2.3,  $A = (\leftarrow, p)_{\preceq_f}$  for some  $p \in X$ . Since  $T \subset A$ , we have that  $p \notin T$  and the zero-dimensionality of  $X$  now implies the existence of a clopen subset  $G \subset X$  such that  $p \in G$  and  $G \cap T = \emptyset$ . Finally, just like before, consider the clopen set

$$B = (\leftarrow, p)_{\preceq_f} \setminus G = (\leftarrow, p]_{\preceq_f} \setminus G$$

which contains  $T$  and is contained in  $V$ . If  $A$  is closed, it is itself clopen and has the same properties as  $B$ . Consequently,  $X$  is normal and strongly zero-dimensional.  $\square$



According to Example 5.4, there exists a zero-dimensional space  $X$  such that  $\mathbb{V}_{\text{cs}}[\mathcal{F}(X)] \neq \emptyset$ , but  $X$  is not selection pointwise-maximal. In view of Theorem 1.5, this suggests the following question.

**Question 5.** Does there exist a normal selection pointwise-maximal space  $X$  which is not strongly zero-dimensional?

According to Theorem 1.1, if  $X$  is a metrizable zero-dimensional space, with  $\mathbb{V}_{\text{cs}}[\mathcal{F}(X)] \neq \emptyset$ , then  $X$  must be a selection pointwise-maximal space. That is, for metrizable spaces, Question 5 is reduced to the following question.

**Question 6.** ([11,14]) Does there exist a zero-dimensional metrizable space  $X$  such that  $\mathbb{V}_{\text{cs}}[\mathcal{F}(X)] \neq \emptyset$  and  $\dim(X) > 0$ ?

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